

# Secondary Theories for étale groupoids

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## Abstract

Generalizing Karoubi's multiplicative K-theory and multiplicative cohomology groups for smooth manifolds we define secondary theories and characteristic classes for smooth étale groupoids. As special cases we obtain versions of the groups of differential characters for smooth étale groupoids and for orbifolds.

## Introduction

In this article we introduce secondary theories and characteristic classes for principal bundles with connections over étale groupoids. In particular, we generalize the multiplicative cohomology groups of Karoubi [K1], [K2] and the groups of Cheeger-Simons differential characters [CS] for smooth manifolds to smooth étale groupoids and arbitrary filtrations of the associated simplicial de Rham complex and analyse their interplay. By specializing to Deligne's 'filtration bête' we obtain generalized versions of Cheeger-Simons differential characters for étale groupoids. In the particular case of a proper étale groupoid representing an orbifold we obtain orbifold versions of these secondary theories, which in the case of differential characters were studied before by Lupercio and Uribe [LU] using the geometric approach of Hopkins and Singer [HS] for the construction of generalized differential cohomology theories.

More generally we are giving a definition of generalized differential characters for smooth étale groupoids associated to an arbitrary filtration of the de Rham complex, which in the case of the 'filtration bête' gives the classical versions of Cheeger and Simons. This more general definition allows to construct an explicit map on the levels of cocycles between multiplicative cohomology and these generalized differential characters. It follows that these general groups of multiplicative cohomology are the natural places for constructing and analysing secondary characteristic classes for principal bundles with connections on étale groupoids. In general such principal bundles, in contrast with the case of smooth manifolds, might not always admit connections, but in the important case for example of principal bundles over proper étale groupoids, which are groupoids representing orbifolds, such connections always exist.

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In the first section we will give an overview of the basic elements of de Rham and Chern-Weil theory for smooth étale groupoids following [LTX]. In the following section we define multiplicative cohomology and generalized differential characters for smooth étale groupoids and study their main properties. We will make explicit use here of the simplicial machinery as developed in [FN], but for the convenience of the reader we will recall many of the details. This was in part influenced by the work of Dupont [D1], [D2] and Dupont-Hain-Zucker [DHZ]. In the third section we introduce multiplicative bundles and versions of multiplicative  $K$ -theory for smooth étale groupoids. In the final section we then construct characteristic classes of elements in multiplicative  $K$ -theory for smooth étale groupoids with values in the groups of multiplicative cohomology and generalized differential characters.

In a sequel to this article we aim to study the relationship between our generalized groups of multiplicative cohomology and smooth Deligne cohomology for étale groupoids and to construct in a unifying way secondary theories and characteristic classes for principal bundles over orbifolds, foliations and differentiable stacks. Versions of smooth Deligne cohomology for particular étale groupoids were also studied before, for example in the case of particular orbifold groupoids by Lupercio-Urbe [LU] and for general transformation groupoids by Gomi [Go]. It will be interesting to analyse if these secondary theories and characteristic classes could be also defined in the context of algebraic geometry, for example in order to extend algebraic differential characters as introduced and studied by Esnault [E1], [E2] to smooth Deligne-Mumford stacks. Real Deligne cohomology groups of proper smooth Deligne-Mumford stacks over the complex numbers  $\mathbb{C}$  appear also in the context of arithmetic intersection theory on Deligne-Mumford stacks as introduced recently by Gillet [Gi].

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## 1 Elements of Chern-Weil theory for étale groupoids

Let us recall the main ingredients of de Rham and Chern-Weil theory for étale groupoids. We will be following here the simplicial approach of [DHZ] and [FN]. For the general theory of étale groupoids and their Chern-Weil theory we refer the reader to [LTX] and [CM1]. We will restrict ourselves in this note to the case of étale groupoids, but many of the concepts and constructions can be derived also in more general contexts.

Let  $\mathcal{G} = [\mathcal{G}_1 \xrightarrow[s]{s} \mathcal{G}_0]$  be a *smooth étale groupoid*, this means that all structure maps of  $\mathcal{G}$

$$\mathcal{G}_1 \times_{t, \mathcal{G}_0, s} \mathcal{G}_1 \xrightarrow{m} \mathcal{G}_1 \xrightarrow{i} \mathcal{G}_1 \xrightarrow[s]{s} \mathcal{G}_0 \xrightarrow{e} \mathcal{G}_1$$

are local diffeomorphisms with  $\mathcal{G}_0, \mathcal{G}_1$  smooth manifolds,  $m$  the composition map,  $i$  the inverse,  $e$  the identity and  $s, t$  the source and target maps.

If in addition, the anchor map

$$(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$$

is *proper*, the groupoid  $\mathcal{G}$  is called a *proper* smooth étale groupoid. Each proper smooth étale groupoid represents an orbifold [M] and therefore the concepts and constructions presented in this note have direct applications to orbifolds.

Associated to any smooth étale groupoid  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  is a canonical smooth simplicial manifold  $\mathcal{G}_\bullet = \{\mathcal{G}_n\}$ , where for each  $n \geq 0$

$$\mathcal{G}_n = \mathcal{G}_1 \times_{t, \mathcal{G}_0, s} \mathcal{G}_1 \cdots \times_{t, \mathcal{G}_0, s} \mathcal{G}_1 \quad (\text{n factors})$$

is the smooth manifold consisting of all composable  $n$ -tuples together with the canonical smooth face and degeneracy maps

$$\varepsilon_i : \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}, \quad \eta_i : \mathcal{G}_n \rightarrow \mathcal{G}_{n+1}$$

for  $0 \leq i \leq n$  such that the usual simplicial identities hold (see [S], [D1]). In other words,  $\mathcal{G}_\bullet$  is a simplicial object in the category of smooth manifolds.

To the simplicial smooth manifold  $\mathcal{G}_\bullet$  we can furthermore associate a simplicial space  $\|\mathcal{G}_\bullet\|$ , its *fat realization*, defined as the quotient space

$$\|\mathcal{G}_\bullet\| = \coprod_{n \geq 0} (\Delta^n \times \mathcal{G}_n) / \sim$$

where the equivalence relation is generated by

$$(\varepsilon^i \times id)(t, x) \sim (id \times \varepsilon_i)(t, x)$$

for any  $(t, x) \in \Delta^{n-1} \times \mathcal{G}_n$ .

A fundamental concept for étale groupoids is that of Morita equivalence. Basically a Morita equivalence class of étale groupoids defines a differentiable stack. Let us very briefly recall the main constructions here. We start with the definition of generalized homomorphisms between étale groupoids.

**Definition 1.1.** A *generalized homomorphism*  $\phi := (Z, \sigma, \tau)$  between étale groupoids  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  and  $\mathcal{H} = [\mathcal{H}_1 \rightrightarrows \mathcal{H}_0]$  is given by a smooth manifold  $Z$ , two smooth maps  $\mathcal{G}_0 \xleftarrow{\tau} Z \xrightarrow{\sigma} \mathcal{H}_0$ , a left action of  $\mathcal{G}_1$  with respect to  $\tau$ , a right action of  $\mathcal{H}_1$  with respect to  $\sigma$ , such that the two actions commute, and  $Z$  is an  $\mathcal{H}_1$ -principal bundle over  $\mathcal{G}_0$ .

Two generalized homomorphisms  $\phi_1 := (Z_1, \sigma_1, \tau_1)$  and  $\phi_2 := (Z_2, \sigma_2, \tau_2)$  from  $\mathcal{G}$  to  $\mathcal{H}$  are called *equivalent* if there is an  $\mathcal{G}_1$ - $\mathcal{H}_1$ -equivariant diffeomorphism  $\psi : Z_1 \rightarrow Z_2$ . Composition of generalized homomorphisms is defined as follows: if  $\phi : \mathcal{G}_0 \xleftarrow{\tau} Z \xrightarrow{\sigma} \mathcal{H}_0$  is a generalized homomorphism from  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  to  $\mathcal{H} = [\mathcal{H}_1 \rightrightarrows \mathcal{H}_0]$  and  $\phi' : \mathcal{H}_0 \xleftarrow{\tau'} Z' \xrightarrow{\sigma'} \mathcal{K}_0$  a generalized homomorphism from  $\mathcal{H} = [\mathcal{H}_1 \rightrightarrows \mathcal{H}_0]$  to  $\mathcal{K} = [\mathcal{K}_1 \rightrightarrows \mathcal{K}_0]$ , then the composition  $\phi \circ \phi' : \mathcal{G}_0 \leftarrow Z'' \rightarrow \mathcal{K}_0$  defined by

$$Z'' = Z \times_{\mathcal{H}_1} Z' := (Z \times_{\sigma, \mathcal{H}_0, \tau'} Z') / (z, z') \sim (zh, h^{-1}z')$$

is a generalized homomorphism from  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  to  $\mathcal{K} = [\mathcal{K}_1 \rightrightarrows \mathcal{K}_0]$ . The composition of equivalence classes of generalized homomorphisms is also associative and we get the *category Grp of étale groupoids*, whose objects are étale groupoids and whose morphisms are generalized homomorphisms. The isomorphisms in this category *Grp* are called *Morita equivalences* and it follows that

any generalized homomorphism of étale groupoids can be decomposed as the composition of a Morita equivalence and a strict homomorphism of groupoids. The category  $\mathcal{G}rp[\mathcal{M}]^{-1}$  obtained from  $\mathcal{G}rp$  via localizing with respect to Morita equivalences gives the *category of differentiable stacks*, i.e. a differentiable stack can be thought of as a Morita equivalence class of an étale groupoid [Pr].

Let us now recall the basic ingredients of de Rham and Chern-Weil theory for étale groupoids using the approach in [DHZ] and [FN] for simplicial smooth manifolds. Working simplicially, there are in fact two versions of the de Rham and singular complexes we can associate to an étale groupoid  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows^s_t \mathcal{G}_0]$  using the associated simplicial smooth manifold  $\mathcal{G}_\bullet$ .

**The de Rham complex of compatible forms.** A simplicial smooth complex  $k$ -form  $\omega$  on  $\mathcal{G}_\bullet$  is a sequence  $\{\omega^{(n)}\}$  of smooth complex  $k$ -forms  $\omega^{(n)} \in \Omega_{dR}^k(\Delta^n \times \mathcal{G}_n)$  satisfying the compatibility condition

$$(\varepsilon^i \times id)^* \omega^{(n)} = (id \times \varepsilon_i)^* \omega^{(n-1)}$$

in  $\Omega_{dR}^k(\Delta^{n-1} \times \mathcal{G}_n)$  for all  $0 \leq i \leq n$  and all  $n \geq 1$ . Let  $\Omega_{dR}^k(\mathcal{G}_\bullet)$  be the set of all simplicial smooth complex  $k$ -forms on  $\mathcal{G}_\bullet$ . The exterior differential on  $\Omega_{dR}^k(\Delta^n \times \mathcal{G}_n)$  induces an exterior differential  $d$  on  $\Omega_{dR}^k(\mathcal{G}_\bullet)$ . We denote by  $(\Omega_{dR}^*(\mathcal{G}_\bullet), d)$  the de Rham complex of compatible forms.

We note that  $(\Omega_{dR}^*(\mathcal{G}_\bullet), d)$  is given as the total complex of a double complex  $(\Omega_{dR}^{*,*}(\mathcal{G}_\bullet), d', d'')$  with

$$\Omega_{dR}^k(\mathcal{G}_\bullet) = \bigoplus_{r+s=k} \Omega_{dR}^{r,s}(\mathcal{G}_\bullet)$$

and  $d = d' + d''$ , where  $\Omega_{dR}^{r,s}(\mathcal{G}_\bullet)$  is the vector space of  $(r+s)$ -forms, which when restricted to  $\Delta^n \times \mathcal{G}_n$  are locally of the form

$$\omega|_{\Delta^n \times \mathcal{G}_n} = \sum a_{i_1 \dots i_r j_1 \dots j_s} dt_{i_1} \wedge \dots \wedge dt_{i_r} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_s},$$

where  $(t_0, \dots, t_n)$  are barycentric coordinates of  $\Delta^n$  and the  $\{x_j\}$  are local coordinates of  $\mathcal{G}_n$ . Furthermore the differentials  $d'$  and  $d''$  are the exterior differentials on  $\Delta^n$  and  $\mathcal{G}_n$  respectively.

Note that  $\omega = \{\omega^{(n)}\}$  gives a smooth  $k$ -form on

$$\coprod_{n \geq 0} (\Delta^n \times \mathcal{G}_n)$$

and the compatible condition is precisely what is needed to define a form on the fat realization  $\|\mathcal{G}_\bullet\|$  of  $\mathcal{G}_\bullet$ .

**The simplicial de Rham complex.** The de Rham complex  $(\mathcal{A}_{dR}^*(\mathcal{G}_\bullet), \delta)$  of  $\mathcal{G}_\bullet$  is given as the total complex of a double complex  $(\mathcal{A}_{dR}^{*,*}(\mathcal{G}_\bullet), \delta', \delta'')$  with

$$\mathcal{A}_{dR}^k(\mathcal{G}_\bullet) = \bigoplus_{r+s=k} \mathcal{A}_{dR}^{r,s}(\mathcal{G}_\bullet)$$

and  $\delta = \delta' + \delta''$ , where  $\mathcal{A}_{dR}^{r,s}(\mathcal{G}_\bullet) = \Omega_{dR}^s(\mathcal{G}_r)$  is the set of smooth complex  $s$ -forms on the smooth manifold  $\mathcal{G}_r$ . Furthermore the differential

$$\delta'' : \mathcal{A}_{dR}^{r,s}(\mathcal{G}_\bullet) \rightarrow \mathcal{A}_{dR}^{r,s+1}(\mathcal{G}_\bullet)$$

is the exterior differential on  $\Omega_{dR}^*(\mathcal{G}_r)$  and the differential

$$\delta' : \mathcal{A}_{dR}^{r,s}(\mathcal{G}_\bullet) \rightarrow \mathcal{A}_{dR}^{r+1,s}(\mathcal{G}_\bullet)$$

is defined as the alternating sum

$$\delta' = \sum_{i=0}^{r+1} (-1)^i \varepsilon_i^*.$$

**The simplicial singular cochain complex.** Given a commutative ring  $R$  we can also associate a singular cochain complex  $(S^*(\mathcal{G}_\bullet; R), \partial)$  to  $\mathcal{G}_\bullet$ . It is defined as a double complex  $(S^{*,*}(\mathcal{G}_\bullet; R), \partial', \partial'')$  with

$$S^k(\mathcal{G}_\bullet; R) = \bigoplus_{r+s=k} S^{r,s}(\mathcal{G}_\bullet; R)$$

and  $\partial = \partial' + \partial''$ , where

$$S^{r,s}(\mathcal{G}_\bullet; R) = S^s(\mathcal{G}_r; R)$$

is the set of singular cochains of degree  $s$  on the smooth manifold  $\mathcal{G}_r$ .

There is an integration map

$$\mathcal{I} : \mathcal{A}_{dR}^{r,s}(\mathcal{G}_\bullet) \rightarrow S^{r,s}(\mathcal{G}_\bullet; \mathbb{C})$$

which gives a morphism of double complexes and Dupont's general version of the de Rham theorem (see [D2], Proposition 6.1) shows that this integration map induces natural isomorphisms

$$H^*(\mathcal{A}_{dR}^*(\mathcal{G}_\bullet, \delta)) \cong H^*(S^*(\mathcal{G}_\bullet, \mathbb{C}), \partial) \cong H^*(\|\mathcal{G}_\bullet\|; \mathbb{C}).$$

From Stokes' theorem it follows that there is also morphism of complexes

$$\mathcal{J} : \Omega_{dR}^*(\mathcal{G}_\bullet, d) \rightarrow (\mathcal{A}_{dR}^*(\mathcal{G}_\bullet), \delta)$$

defined on  $\Omega_{dR}^*(\Delta^n \times \mathcal{G}_n)$  by integration over the simplex  $\Delta^n$

$$\omega^{(n)} \in \Omega_{dR}^*(\Delta^n \times \mathcal{G}_n) \mapsto \int_{\Delta^n} \omega^{(n)}.$$

This morphism is in fact a quasi-isomorphism (see [D1], Theorem 2.3 and Corollary 2.8), i.e. we have

$$H^*(\Omega_{dR}^*(\mathcal{G}_\bullet, d)) \cong H^*(\mathcal{A}_{dR}^*(\mathcal{G}_\bullet), \delta) \cong H^*(\|\mathcal{G}_\bullet\|; \mathbb{C}).$$

**The singular cochain complex of compatible cochains.** Let  $R$  be a commutative ring. A compatible singular cochain  $c$  on  $\mathcal{G}_\bullet$  is a sequence  $\{c^{(n)}\}$  of cochains  $c^{(n)} \in S^k(\Delta^n \times \mathcal{G}_n; R)$  satisfying the compatibility condition

$$(\varepsilon^i \times id)^* c^{(n)} = (id \times \varepsilon_i)^* c^{(n-1)}$$

in  $S^k(\Delta^{n-1} \times \mathcal{G}_n)$  for all  $0 \leq i \leq n$  and all  $n \geq 1$ . Let  $C^k(\mathcal{G}_\bullet; R)$  be the set of all compatible singular cochains on  $\mathcal{G}_\bullet$  and  $(C^*(\mathcal{G}_\bullet; R), d)$  be the singular cochain complex of compatible cochains.

The natural inclusion of cochain complexes

$$(C^*(\mathcal{G}_\bullet; R), d) \rightarrow (S^*(\mathcal{G}_\bullet; R), \partial)$$

is a quasi-isomorphism [DHZ].

Integrating forms preserves the compatibility conditions and therefore we get an induced map of complexes [DHZ]

$$\mathcal{I}' : \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C})$$

fitting into a commutative diagram

$$\begin{array}{ccc} \Omega_{dR}^*(\mathcal{G}_\bullet) & \xrightarrow{\mathcal{I}'} & C^*(\mathcal{G}_\bullet; \mathbb{C}) \\ \downarrow \mathcal{J} & & \downarrow \\ A_{dR}^*(\mathcal{G}_\bullet) & \xrightarrow{\mathcal{I}} & S^*(\mathcal{G}_\bullet; \mathbb{C}) \end{array}$$

and which is again a quasi-isomorphism, i.e. we have

$$H^*(\Omega_{dR}^*(\mathcal{G}_\bullet), d) \cong H^*(C^*(\mathcal{G}_\bullet; \mathbb{C}), \partial).$$

We call  $H_{dR}^*(\mathcal{G}_\bullet) := H^*(\Omega_{dR}^*(\mathcal{G}_\bullet), d)$  the *de Rham cohomology* of the groupoid  $\mathcal{G}$  and  $H^*(\mathcal{G}_\bullet, R) := H^*(C^*(\mathcal{G}_\bullet; R), \partial)$  the *singular cohomology* of  $\mathcal{G}$ . De Rham and singular cohomology behave well with respect to Morita equivalence of étale groupoids and are therefore well-defined invariants for differentiable stacks (see [B1], Def. 9. or [LTX], 3.1).

**Proposition 1.2.** *Let  $R$  be a commutative ring. If  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  and  $\mathcal{H} = [\mathcal{H}_1 \rightrightarrows \mathcal{H}_0]$  are étale groupoids which are Morita equivalent, then we have isomorphisms*

$$H_{dR}^k(\mathcal{G}_\bullet) \xrightarrow{\cong} H_{dR}^k(\mathcal{H}_\bullet), \text{ and } H^k(\mathcal{G}_\bullet, R) \xrightarrow{\cong} H^k(\mathcal{H}_\bullet, R).$$

*Proof.* That the simplicial de Rham and singular complexes of Morita equivalent étale groupoids give isomorphic cohomology groups is well-known (see for example [TXL], Prop. 2.15, [B1] or [CM1]). The above quasi-isomorphisms between the respective complexes of simplicial and compatible forms and cochains then give the desired result.  $\square$

**Example 1.3.** If  $\mathcal{G} = [M \rightrightarrows M]$  is just a smooth manifold  $M$ , then the de Rham cohomology and singular cohomology groups of  $\mathcal{G}$  are just the ones for smooth manifolds. Indeed, if  $\{U_i\}$  is an open covering of  $M$  and  $\coprod_i U_i \rightarrow M$  the étale map, then the étale groupoid  $\Gamma = [\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i]$  is Morita equivalent to  $\mathcal{G}$  and  $H_{dR}^*(\Gamma_\bullet) \cong H_{dR}^*(M)$ .

**Example 1.4.** Let  $G$  be a compact Lie group. If  $\mathcal{G} = [G \times M \rightrightarrows M]$  is the transformation groupoid, then  $H_{dR}^*(\mathcal{G}_\bullet)$  and  $H^*(\mathcal{G}_\bullet, R)$  are the  $G$ -equivariant de Rham and Borel cohomology groups of  $M$ .

**Example 1.5.** Let  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  be a proper étale groupoid representing an orbifold and let  $\Lambda(\mathcal{G})$  be the associated inertia groupoid, i. e. the transformation groupoid  $\Lambda(\mathcal{G}) = [\Omega(\mathcal{G}) \rtimes \mathcal{G} \rightrightarrows \Omega(\mathcal{G})]$  in which we denote by  $\Omega(\mathcal{G}) = \{g \in \mathcal{G}_1 \mid s(g) = t(g)\}$  the manifold of closed loops. Then  $H_{dR}^*(\Lambda(\mathcal{G})_\bullet)$  and  $H^*(\Lambda(\mathcal{G})_\bullet, R)$  are the associated orbifold cohomology groups.

In these examples we could have either used the simplicial or compatible complexes of  $\mathcal{G}_\bullet$ . We will use the compatible de Rham and cochain complexes to define generalized versions of Karoubi's multiplicative cohomology and Cheeger-Simons differential characters for a given étale groupoid  $\mathcal{G}$ .

Finally we recall the basic aspects of Chern-Weil theory for étale groupoids as developed in [LTX].

**Principal  $G$ -bundles over étale groupoids.** Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$  and  $\mathcal{G} = [\mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0]$  be an étale groupoid. A *principal  $G$ -bundle* over  $\mathcal{G}$  consists of a (right)  $G$ -bundle  $P \xrightarrow{\pi} \mathcal{G}_0$  over the smooth manifold  $\mathcal{G}_0$  such that the groupoid  $\mathcal{G}$  acts on  $P \xrightarrow{\pi} \mathcal{G}_0$  and this action commutes with the  $G$ -action (see [LTX], Def. 2.2).

It turns out, that the category of principal  $G$ -bundles over an étale groupoid  $\mathcal{G} = [\mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0]$  is equivalent to the category of principal  $G$ -bundles over the simplicial smooth manifold  $\mathcal{G}_\bullet$  (see [LTX], Prop. 2.4). Here a principal  $G$ -bundle over  $\mathcal{G}_\bullet$  is given by a simplicial smooth manifold  $P_\bullet$  and a morphism  $\pi_\bullet : P_\bullet \rightarrow \mathcal{G}_\bullet$  of simplicial smooth manifolds, such that

- (i) for each  $n$  the map  $\pi_p : P_n \rightarrow \mathcal{G}_n$  is a principal  $G$ -bundle over  $\mathcal{G}_n$
- (ii) for each morphism  $f : [m] \rightarrow [n]$  of the simplex category  $\Delta$  the induced map  $f^* : P_n \rightarrow P_m$  is a morphism of  $G$ -bundles, i.e. we have a commutative diagram

$$\begin{array}{ccc} P_n & \xrightarrow{f^*} & P_m \\ \downarrow & & \downarrow \\ \mathcal{G}_n & \xrightarrow{f^*} & \mathcal{G}_m \end{array}$$

The last condition is just saying that the degeneracy and face maps are morphisms of principal  $G$ -bundles.

It follows also, that if  $\pi_\bullet : P_\bullet \rightarrow \mathcal{G}_\bullet$  is a principal  $G$ -bundle over  $\mathcal{G}_\bullet$ , then  $|\pi_\bullet| : |P_\bullet| \rightarrow |\mathcal{G}_\bullet|$  is a principal  $G$ -bundle with  $G$ -action induced by

$$\Delta^n \times P_n \times G \rightarrow \Delta^n \times P_n, (t, x, g) \mapsto (t, xg).$$

It can be shown that the category of principal  $G$ -bundles over an étale groupoid  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  is equivalent to the category of generalized homomorphisms from  $[\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  to the groupoid  $[G \rightrightarrows \cdot]$  and therefore if  $\mathcal{G}$  and  $\mathcal{G}'$  are Morita equivalent étale groupoids, there is an equivalence of categories of principal  $G$ -bundles over  $\mathcal{G}$  and  $\mathcal{G}'$  (see [LTX], Prop. 2.11 and Cor 2.12). There is also a well-defined notion of pull-back of principal  $G$ -bundles over étale groupoids along generalized homomorphisms.

**Connections and curvature for principal  $G$ -bundles.** Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Furthermore let  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  be an étale groupoid and  $P \xrightarrow{\pi} \mathcal{G}_0$  be a principal  $G$ -bundle over  $\mathcal{G}$ . Following [LTX], 3.2 we consider pseudo-connection and pseudo-curvature forms for principal  $G$ -bundles over étale groupoids. A *pseudo-connection* is a connection 1-form  $\theta \in \Omega_{dR}^1(P) \otimes \mathfrak{g}$

of the  $G$ -bundle  $P \xrightarrow{\pi} \mathcal{G}_0$  (ignoring the groupoid action). The *total pseudo-curvature* is a 2-form  $\Omega_{\text{total}} \in \Omega_{dR}^2(Q_\bullet) \otimes \mathfrak{g}$  defined as

$$\Omega_{\text{total}} = \delta\theta + \frac{1}{2}[\theta, \theta] = \partial\theta + \Omega$$

where  $\Omega = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega_{dR}^2(P) \otimes \mathfrak{g}$  is the curvature form corresponding to  $\theta$  and  $Q = \mathcal{G}_1 \times_{s, \mathcal{G}_0, \pi} P$ . The total pseudo-curvature  $\Omega_{\text{total}}$  has therefore two terms. The first term  $\partial\theta := s^*\theta - t^*\theta$  is a  $\mathfrak{g}$ -valued 1-form on  $Q$  and the second term  $\Omega$  is a  $\mathfrak{g}$ -valued 2-form on  $P$ . Both terms are of total degree 2 in the double complex  $\Omega^*(Q_\bullet) \otimes \mathfrak{g}$ . As for connections on principal  $G$ -bundles on smooth manifolds the pseudo-curvature also satisfies a Bianchi type identity (see [LTX], Prop. 3.3).

A pseudo-connection  $\theta \in \Omega_{dR}^1(P) \otimes \mathfrak{g}$  is called a *connection* if  $\partial\theta = 0$  and in this case  $\Omega_{\text{total}} = \Omega \in \Omega_{dR}^2(P) \otimes \mathfrak{g}$  is called the *curvature*. A connection  $\theta$  is said to be a *flat connection* if its curvature  $\Omega$  vanishes. A necessary and sufficient condition for a pseudo-connection  $\theta$  to be a connection is that  $\theta$  is a basic form with respect to the action of the pseudo-group of local bisections of the groupoid  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  (see [LTX], Prop. 3.6).

In general, in contrast to pseudo-connections, connections for principal  $G$ -bundles over étale groupoids might not always exist as is shown by the example of the groupoid  $[G \rightrightarrows \cdot]$ , where  $G$  is a Lie group. The map  $G \rightarrow \cdot$  can be considered as a principal  $G$ -bundle over  $[G \rightrightarrows \cdot]$ , where the groupoid acts on  $G$  by left translations. But connections can only exist if  $G$  is a discrete group (see [LTX], Example 3.12).

As for manifolds connections behave well with respect to pull-backs under generalized homomorphisms of étale groupoids. If  $\mathcal{G}$  and  $\mathcal{G}'$  are Morita equivalent groupoids, there is an equivalence of categories of principal  $G$ -bundles with connections over  $\mathcal{G}$  and  $\mathcal{G}'$ . Similarly there is an equivalence of categories of principal  $G$ -bundles with flat connections over  $\mathcal{G}$  and  $\mathcal{G}'$ . Therefore, one can also speak of connections and flat connections of principal  $G$ -bundles over differentiable stacks, though connections might not always exist. The groupoid example above corresponds to the classifying stack  $\mathcal{B}G$  of the Lie group  $G$  and so principal  $G$ -bundles over  $\mathcal{B}G$  don't have connections unless  $G$  is a discrete group. But we note the following important case where connections do exist [LTX], Theorem 3.16:

**Proposition 1.6.** *Let  $G$  be a Lie group. Any principal  $G$ -bundle over a proper étale groupoid  $\mathcal{G}$  admits a connection. In particular, principal  $G$ -bundles over orbifolds admit a connection.*

Using the correspondence between the category of principal  $G$ -bundles over an étale groupoid  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  with the category of principal  $G$ -bundles over the associated simplicial smooth manifold  $\mathcal{G}_\bullet$ , a connection  $\theta$  on a principal  $G$ -bundle over  $\mathcal{G}$  corresponds to a connection, also denoted by  $\theta$ , on the principal  $G$ -bundle  $\pi_\bullet : P_\bullet \rightarrow \mathcal{G}_\bullet$  over the associated simplicial smooth manifold  $\mathcal{G}_\bullet$ , i. e. a  $G$ -invariant 1-form in the de Rham complex of compatible forms

$$\theta \in \Omega_{dR}^1(P_\bullet) \otimes \mathfrak{g}$$

taking values in the Lie algebra  $\mathfrak{g}$  of  $G$ , on which  $G$  acts via the adjoint representation, such that for each  $n$  the restriction

$$\theta^{(n)} = \theta|_{\Delta^n \times P_n},$$



is a connection on the bundle  $\pi_n : \Delta^n \times P_n \rightarrow \Delta^n \times \mathcal{G}_n$ . In this way  $\theta = \{\theta^{(n)}\}$  can as well be interpreted as a sequence of  $\mathfrak{g}$ -valued compatible 1-forms.

The curvature  $\Omega$  of a connection form  $\theta$  is then given as the differential form

$$\Omega = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega_{dR}^2(\mathcal{G}_\bullet) \otimes \mathfrak{g}.$$

The behaviour of connection forms under invariant polynomials with respect to the Chern-Weil map for the associated simplicial smooth manifold is given as follows (see [D1], Proposition 3.7)

**Theorem 1.7.** *Let  $G$  be a Lie group and  $\Phi$  be an invariant polynomial. The differential form  $\Phi(\theta) \in \Omega_{dR}^*(P_\bullet)$  is a closed form and descends to a closed form in  $\Omega_{dR}^*(\mathcal{G}_\bullet)$  and its cohomology class represents the image of the class  $\Phi \in H^*(BG; \mathbb{C})$  under the Chern-Weil map  $H^*(BG; \mathbb{C}) \rightarrow H^*(||\mathcal{G}_\bullet||; \mathbb{C})$  associated to the principal  $G$ -bundle  $\pi_\bullet : P_\bullet \rightarrow \mathcal{G}_\bullet$ , where  $BG$  is the classifying space of  $G$ .*

We will denote the associated form in  $\Omega_{dR}^*(\mathcal{G}_\bullet)$  also by  $\Phi(\theta)$ . When defining characteristic classes it is necessary that given any connection on a principal bundle, we can construct a connection on (a model of) the universal bundle that pulls back to the given one. This follows from the following theorem:

**Theorem 1.8.** *Let  $G$  be a Lie group and  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  be an étale groupoid. Let  $P \xrightarrow{\pi} \mathcal{G}_0$  be a principal  $G$ -bundle over  $\mathcal{G}$  with connection. Then there exists a bisimplicial smooth manifold  $B_{\bullet\bullet}$  of the homotopy type of the classifying space  $BG$  and a  $G$ -principal bundle  $U_{\bullet\bullet} \rightarrow B_{\bullet\bullet}$  with a connection  $\theta_{U_{\bullet\bullet}} \in \Omega_{dR}^1(U_{\bullet\bullet}; \mathfrak{g})$  and a morphism  $(\Psi, \psi)$  of  $G$ -bundles*

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\Psi} & U_{\bullet\bullet} \\ \downarrow & & \downarrow \\ \mathcal{G}_\bullet & \xrightarrow{\psi} & B_{\bullet\bullet} \end{array}$$

such that  $\Psi^*(\theta_{U_{\bullet\bullet}}) = \theta$ .

*Proof.* This is basically [FN], Theorem 1.2, (for principal  $GL_n(\mathbb{C})$ -bundles this can also be found in [DHZ], Prop. 6.15.) using Chern-Weil theory for bisimplicial smooth manifolds and the categorical correspondence between principal  $G$ -bundles over an étale groupoid  $\mathcal{G}$  and principal  $G$ -bundles over the associated simplicial smooth manifold  $\mathcal{G}_\bullet$ .  $\square$

## 2 Multiplicative Cohomology and Differential Characters for étale groupoids

In this section we will construct general versions of Karoubi's multiplicative cohomology and Cheeger-Simons differential characters for étale groupoids with respect to any given filtration of the de Rham complex. We will closely follow the approach for simplicial smooth manifolds as described in [FN] and for the convenience of the reader will recall the necessary constructions and concepts. As a special case with respect to the 'filtration bête' we will also recover the

group of Cheeger-Simons differential characters for étale groupoids as discussed in [LU].

For a given complex of abelian groups  $C^*$  let  $\sigma_{\geq p}C^*$  denote the filtration via truncation in degrees below  $p$  and similarly let  $\sigma_{< p}C^*$  denote the truncation of  $C^*$  in degrees greater or equal  $p$ . We will first consider the special case of Deligne's 'filtration bête' [De] for the de Rham complex  $\Omega_{dR}^*(\mathcal{G}_\bullet)$  of a smooth étale groupoid  $\mathcal{G}$ . The 'filtration bête'  $\sigma = \{\sigma_{\geq p}\Omega_{dR}^*(\mathcal{G}_\bullet)\}$  is given via truncation in degrees below  $p$

$$\sigma_{\geq p}\Omega_{dR}^j(\mathcal{G}_\bullet) = \begin{cases} 0 & j < p, \\ \Omega_{dR}^j(\mathcal{G}_\bullet) & j \geq p \end{cases}$$

We define the group of Cheeger-Simons differential characters as follows:

**Definition 2.1.** Let  $\mathcal{G}$  be a smooth étale groupoid and  $\Lambda$  be a subgroup of  $\mathbb{C}$ . The group of (mod  $\Lambda$ ) differential characters of degree  $k$  of  $\mathcal{G}$  is given by

$$\hat{H}^{k-1}(\mathcal{G}_\bullet; \mathbb{C}/\Lambda) = H^k(\text{cone}(\sigma_{\geq k}\Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}/\Lambda))).$$

Now let  $\mathcal{F} = \{F^r\Omega_{dR}^*(\mathcal{G}_\bullet)\}$  be any given filtration of the de Rham complex. We define the multiplicative cohomology groups of  $\mathcal{G}$  with respect to  $\mathcal{F}$  as follows:

**Definition 2.2.** Let  $\mathcal{G}$  be a smooth étale groupoid,  $\Lambda$  be a subgroup of  $\mathbb{C}$  and  $\mathcal{F} = \{F^r\Omega_{dR}^*(\mathcal{G}_\bullet)\}$  be a filtration of  $\Omega_{dR}^*(\mathcal{G}_\bullet)$ . The groups of multiplicative cohomology of  $\mathcal{G}$  associated to the filtration  $\mathcal{F}$  are given by

$$MH_n^{2r}(\mathcal{G}_\bullet; \Lambda; \mathcal{F}) = H^{2r-n}(\text{cone}(C^*(\mathcal{G}_\bullet; \Lambda) \oplus F^r\Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}))).$$

If the étale groupoid  $\mathcal{G} = [M \rightrightarrows M]$  is just a smooth manifold  $M$  as in Example 1.3, we recover the multiplicative cohomology groups of Karoubi [K1], [K2] and we have  $MH_n^{2r}(M; \Lambda; \mathcal{F}) \cong MH_n^{2r}(\mathcal{G}_\bullet; \Lambda; \mathcal{F})$ . If  $G$  is a compact Lie group and  $\mathcal{G} = [G \times M \rightrightarrows M]$  a transformation groupoid as in Example 1.4, we will get equivariant versions of multiplicative cohomology. If  $\mathcal{G}$  is a proper étale groupoid representing an orbifold as in Example 1.5, then the groups of multiplicative cohomology of the associated inertia groupoid  $MH_n^{2r}(\Lambda(G)_\bullet; \Lambda; \mathcal{F})$  are the corresponding orbifold versions of multiplicative cohomology.

Using similar arguments as in [TXL], 2.1 it is possible to show that the groups of multiplicative cohomology of étale groupoids behave well with respect to generalized homomorphisms and Morita equivalence and under mild conditions therefore will give interesting invariants for differentiable stacks and orbifolds, which we aim to study in detail in a follow-up article. A main aspect here will be to relate them to versions of smooth Deligne cohomology for differentiable stacks and orbifolds.

We also introduce here a more general version of differential characters for étale groupoids associated to any given filtration of the de Rham complex. For smooth manifolds these invariants were studied systematically by the first author in [F].

**Definition 2.3.** Let  $\mathcal{G}$  be a smooth étale groupoid,  $\Lambda$  be a subgroup of  $\mathbb{C}$  and  $\mathcal{F} = \{F^r\Omega_{dR}^*(\mathcal{G}_\bullet)\}$  be a filtration of  $\Omega_{dR}^*(\mathcal{G}_\bullet)$ . The groups of differential characters (mod  $\Lambda$ ) of degree  $k$  of  $\mathcal{G}$  associated to the filtration  $\mathcal{F}$  are given by

$$\hat{H}_r^{k-1}(\mathcal{G}_\bullet; \mathbb{C}/\Lambda; \mathcal{F}) = H^k(\text{cone}(\sigma_{\geq k}F^r\Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}/\Lambda))).$$

If  $\mathcal{F}$  is Deligne's 'filtration bête' of  $\Omega_{dR}^*(\mathcal{G}_\bullet)$ , we recover the ordinary groups of differential characters of  $\mathcal{G}$  as in Definition 2.1. And again as with the more general groups of multiplicative cohomology, looking at the étale groupoids in Examples 1.3-1.5 we will recover the classical Cheeger-Simons differential characters [CS] in the case of smooth manifolds, equivariant versions in the case of a transformation groupoid [Ge] and in the case of a proper étale groupoid orbifold versions of differential characters associated to the inertia groupoid [LU].

The following theorem generalizes [F], Theorem 2.3 for smooth manifolds:

**Theorem 2.4.** *Let  $\mathcal{G}$  be a smooth étale groupoid,  $\Lambda$  be a subgroup of  $\mathbb{C}$  and  $\mathcal{F} = \{F^r \Omega_{dR}^*(\mathcal{G}_\bullet)\}$  be a filtration of  $\Omega_{dR}^*(\mathcal{G}_\bullet)$ . There exists a surjective map*

$$\Xi : \hat{H}_r^{2r-n-1}(\mathcal{G}_\bullet; \mathbb{C}/\Lambda; \mathcal{F}) \rightarrow MH_n^{2r}(\mathcal{G}_\bullet; \Lambda; \mathcal{F})$$

whose kernel is the group of forms in  $F^r \Omega_{dR}^{2r-n-1}(\mathcal{G}_\bullet)$  modulo those forms that are closed and whose complex cohomology class is the image of a class in  $H^*(\mathcal{G}_\bullet; \Lambda)$ .

**Proof:** We denote by  $\mathcal{A}(F^r)$  and  $\mathcal{B}(F^r)$  the cone complexes used in the definition of the groups of differential characters and multiplicative cohomology associated to the filtration  $\mathcal{F}$ , i.e.

$$\mathcal{A}(F^r) = \text{cone}(\sigma_{\geq k} F^r \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}/\Lambda))$$

$$\mathcal{B}(F^r) = \text{cone}(C^*(\mathcal{G}_\bullet; \Lambda) \oplus F^r \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}))$$

There is a quasi-isomorphism between the cone complexes

$$\text{cone}(\sigma_{\geq k} F^r \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}/\Lambda))$$

$$\text{cone}(C^*(\mathcal{G}_\bullet; \Lambda) \oplus \sigma_{\geq k} F^r \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}))$$

and we get a short exact sequence of complexes

$$0 \rightarrow \mathcal{A}(F^r) \rightarrow \mathcal{B}(F^r) \rightarrow \sigma_{< k} F^r \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow 0$$

where  $\sigma_{< k}$  denotes truncation in degrees greater or equal to  $k$ . Everything follows now from the long exact sequence in cohomology associated to this short exact sequences of complexes, because for  $k = 2r - n$  the cohomology group  $H^{2r-n}(\sigma_{< 2r-n} F^r \Omega_{dR}^*(\mathcal{G}_\bullet))$  is trivial.  $\square$

We can also identify the Cheeger-Simons differential characters with multiplicative cohomology groups in the following way

**Corollary 2.5.** *Let  $\mathcal{G}$  be a smooth étale groupoid and  $\Lambda$  be a subgroup of  $\mathbb{C}$ . There is an isomorphism*

$$\hat{H}^{r-1}(\mathcal{G}_\bullet; \mathbb{C}/\Lambda) \cong MH_r^{2r}(\mathcal{G}_\bullet; \Lambda; \sigma)$$

**Proof.** This is a direct consequence of Theorem 2.4 for the case  $n = r$  and the filtration  $\mathcal{F}$  is just Deligne's 'filtration bête'  $\square$

Similarly as for smooth manifolds, it can be shown that the multiplicative cohomology groups for smooth étale groupoids fit into a long exact sequence (compare [K3]).

$$\begin{aligned} \dots \rightarrow H^{2r-n-1}(\|\mathcal{G}_\bullet\|; \Lambda) \rightarrow H^{2r-n-1}(\Omega_{dR}^*(\mathcal{G}_\bullet/F^r \Omega_{dR}^*(\mathcal{G}_\bullet))) \rightarrow \\ \rightarrow MH_n^{2r}(\mathcal{G}_\bullet; \Lambda; \mathcal{F}) \rightarrow \dots \end{aligned}$$

Following from the above it can be shown as in [CS] that the groups of differential characters also fit into short exact sequences.

### 3 Multiplicative K-theory for étale groupoids

Let  $\mathcal{G} = [\mathcal{G}_1 \rightrightarrows \mathcal{G}_0]$  be an étale groupoid. Further let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Assume we are given a principal  $G$ -bundle on  $\mathcal{G}$  with connection forms  $\theta_0, \dots, \theta_q$  giving associated connection forms  $\theta_0, \dots, \theta_q$  on the principal  $G$ -bundle  $\pi_\bullet : P_\bullet \rightarrow \mathcal{G}_\bullet$ , i.e.

$$\theta_j \in \Omega_{dR}^1(P_\bullet) \otimes \mathfrak{g}$$

such that for all  $p$  and all  $0 \leq j \leq q$

$$\theta_j^{(p)} \in \Omega_{dR}^1(\Delta^p \times P_p) \otimes \mathfrak{g}$$

i.e. the restrictions  $\theta_j^{(p)}$  are connections on the bundle

$$\Delta^p \times P_p \rightarrow \Delta^p \times \mathcal{G}_p.$$

Fix  $q$  and let  $\Delta^q$  be the standard simplex in  $\mathbb{R}^{q+1}$  parametrized by coordinates  $(s_0, \dots, s_q)$ .

**Lemma 3.1.** *The form  $\sum_{j=0}^q \theta_j s_j$  defines a connection on the pullback bundle  $\pi^* P_\bullet \rightarrow \mathcal{G}_\bullet \times \Delta^q$ , where  $\pi : \mathcal{G}_\bullet \times \Delta^q \rightarrow \mathcal{G}_\bullet$  is the projection.*

**Proof.** For each  $m$  the sum  $(\sum_{j=0}^q \theta_j s_j)^{(m)} = \sum_{j=0}^q \theta_j^{(m)} s_j$  is a connection on the bundle

$$\Delta^m \times P_m \times \Delta^q \rightarrow \Delta^m \times \mathcal{G}_m \times \Delta^q.$$

We have to verify that the compatibility conditions hold. The strict simplicial structure on  $\mathcal{G}_\bullet \times \Delta^q$  is given by the maps  $\varepsilon'_i = \varepsilon_i \times id_{\Delta^q}$  for all  $i$ , where  $\varepsilon_i$  is the map given by the simplicial structure on  $\mathcal{G}_\bullet$ . We have

$$(\varepsilon^i \times id_{\mathcal{G}_m \times \Delta^q})^* (\sum_{j=0}^q \theta_j^{(m)} s_j) = \sum_{j=0}^q (\varepsilon^i \times id_{\mathcal{G}_m})^* \theta_j^{(m)} s_j$$

since the forms  $\theta_j^{(m)} s_j$  are in

$$\Omega_{dR}^1(\Delta^m \times P_m; \mathfrak{g}) \otimes \Omega_{dR}^0(\Delta^q) \subset \Omega_{dR}^1(\Delta^m \times P_m \times \Delta^q).$$

Now, since the  $\theta_j$  satisfy the compatibility conditions we have

$$\sum_{j=0}^q (\varepsilon^i \times id_{\mathcal{G}_m})^* \theta_j^{(m)} s_j = \sum_{j=0}^q (id_{\Delta^{m-1}} \times \varepsilon_i)^* \theta_j^{(m-1)} s_j.$$

As before we have

$$\sum_{j=0}^q (id_{\Delta^{m-1}} \times \varepsilon_i)^* \theta_j^{(m-1)} s_j = (id_{\Delta^{m-1}} \times \varepsilon'_i)^* (\sum_{j=0}^q \theta_j^{(m-1)} s_j),$$

which proves the lemma □

Given an invariant polynomial  $\Phi$  of degree  $k$ , we denote by

$$\tilde{\Theta}_q(\Phi; \theta_0, \dots, \theta_q) \in \Omega_{dR}^{2k}(\mathcal{G}_\bullet \times \Delta^q)$$

the characteristic form on  $\mathcal{G}_\bullet$  associated to  $\Phi$  for the curvature of the connection  $\sum_{j=0}^q \theta_j s_j$ . Whenever  $\Phi$  is understood, we will omit it from the notation for the above form. The closed form  $\tilde{\Theta}_q(\theta_0, \dots, \theta_q)$  is a family of compatible closed forms

$$\tilde{\Theta}_q^{(m)}(\theta_0, \dots, \theta_q) \in \Omega_{dR}^{2k}(\Delta^m \times \mathcal{G}_m \times \Delta^q).$$

We define a form  $\Theta_q(\theta_0, \dots, \theta_q) \in \Omega_{dR}^{2k-q}(\mathcal{G}_\bullet)$  by integration

$$\Theta_q(\theta_0, \dots, \theta_q) = \int_{\Delta^q} \tilde{\Theta}_q(\theta_0, \dots, \theta_q),$$

i.e.  $\Theta_q(\theta_0, \dots, \theta_q)$  is the family of forms

$$\Theta_q^{(m)}(\theta_0, \dots, \theta_q) = \int_{\Delta^q} \tilde{\Theta}_q^{(m)}(\theta_0, \dots, \theta_q).$$

These forms satisfy the compatibility conditions since the diagram

$$\begin{array}{ccc} \Omega_{dR}^*(\Delta^m \times \mathcal{G}_m \times \Delta^q) & \xrightarrow{\int_{\Delta^q}} & \Omega_{dR}^*(\Delta^m \times \mathcal{G}_m) \\ (\varepsilon^i \times \text{id}_{\mathcal{G}_m \times \Delta^q})^* \downarrow & & \downarrow (\varepsilon^i \times \text{id}_{\mathcal{G}_m})^* \\ \Omega_{dR}^*(\Delta^{m-1} \times \mathcal{G}_m \times \Delta^q) & \xrightarrow{\int_{\Delta^q}} & \Omega_{dR}^*(\Delta^{m-1} \times \mathcal{G}_m) \\ (id_{\Delta^{m-1}} \times \varepsilon'_i)^* \uparrow & & \uparrow (id_{\Delta^{m-1}} \times \varepsilon_i)^* \\ \Omega_{dR}^*(\Delta^{m-1} \times \mathcal{G}_{m-1} \times \Delta^q) & \xrightarrow{\int_{\Delta^q}} & \Omega_{dR}^*(\Delta^{m-1} \times \mathcal{G}_{m-1}) \end{array}$$

commutes and the forms  $\tilde{\Theta}_q^{(m)}(\theta_0, \dots, \theta_q) \in \Omega_{dR}^{2k}(\Delta^m \times \mathcal{G}_m \times \Delta^q)$  are compatible.

If we denote by  $t$  the variables on the simplices  $\Delta^p$ , by  $x$  the variables on the smooth manifolds  $\mathcal{G}_p$  and by  $s$  the variables on the simplex  $\Delta^q$ , then we can write the differential on the complex  $\Omega_{dR}^*(\Delta^q \times \mathcal{G}_\bullet)$  as  $d = d_s + d_{t,x}$ , where  $d_{t,x}$  is the differential of the complex  $\Omega_{dR}^*(\mathcal{G}_\bullet)$ . Since  $\tilde{\Theta}_q(\theta_0, \dots, \theta_q)$  is closed, we have

$$d_{t,x} \tilde{\Theta}_q(\theta_0, \dots, \theta_q) = -d_s \tilde{\Theta}_q(\theta_0, \dots, \theta_q).$$

Then we have

$$\begin{aligned} d_{t,x} \Theta_q(\theta_0, \dots, \theta_q) &= d_{t,x} \int_{\Delta^q} \tilde{\Theta}_q(\theta_0, \dots, \theta_q) = \\ &= \int_{\Delta^q} d_{t,x} \tilde{\Theta}_q(\theta_0, \dots, \theta_q) = - \int_{\Delta^q} d_s \tilde{\Theta}_q(\theta_0, \dots, \theta_q). \end{aligned}$$

By Stokes' theorem the last integral is equal to  $-\int_{\partial \Delta^q} \tilde{\Theta}_q(\theta_0, \dots, \theta_q)$ , so we have proven the analogue of Theorem 3.3 in [K2] for smooth étale groupoids:

**Proposition 3.2.** *Let  $\mathcal{G}$  be a smooth étale groupoid. In the de Rham complex  $\Omega_{dR}^*(\mathcal{G}_\bullet)$  of compatible forms we have*

$$d\Theta_q(\theta_0, \dots, \theta_q) = - \sum_{i=0}^q (-1)^i \Theta_{q-1}(\theta_0, \dots, \hat{\theta}_i, \dots, \theta_q).$$

In particular, for  $q = 1$  we have that if given any two connections  $\theta_0$  and  $\theta_1$  on  $P_\bullet$  and an invariant polynomial  $\Phi$ , we can write canonically

$$\Phi(\theta_1) - \Phi(\theta_0) = d\Theta_1(\Phi; \theta_0, \theta_1).$$

Write a formal series of invariant polynomials  $\Phi$  as a sum  $\sum_r \Phi_r$  with  $\Phi_r$  a homogeneous polynomial of degree  $r$  (see [K2],[K3]). Let  $\mathcal{F} = \{F^r \Omega_{dR}^*(\mathcal{G}_\bullet)\}$  be again a filtration of the de Rham complex of  $\mathcal{G}_\bullet$  and

$$\omega = \sum_r \omega_r, \quad \eta = \sum_r \eta_r$$

be formal sums of forms in  $\Omega_{dR}^*(\mathcal{G}_\bullet)$  (note that we do not require that  $\omega_r$  is of degree  $r$ , actually most of the times this will not be the case). We will write  $\omega = \eta \bmod \mathcal{F}$  if and only if for each  $r$  we have

$$\omega_r - \eta_r \in F^r \Omega_{dR}^*(\mathcal{G}_\bullet). \quad (3.1)$$

We will also write  $\omega = \eta \bmod \tilde{\mathcal{F}}$  when for each  $r$  the above equation is satisfied modulo exact forms. We will not distinguish between an invariant polynomial or a formal series in what follows, writing just  $\Phi$  and  $\omega$  also for formal sums. Let us define the notion of a multiplicative bundle:

**Definition 3.3.** Let  $\mathcal{G}$  be a smooth étale groupoid and  $\Phi$  be an invariant polynomial (or a formal series) and  $\mathcal{F} = \{F^r \Omega_{dR}^*(\mathcal{G}_\bullet)\}$  be a filtration of the de Rham complex of  $\mathcal{G}$ . An  $(\mathcal{F}, \Phi)$ -multiplicative bundle over  $\mathcal{G}$  is a triple  $(P_\bullet, \theta, \omega)$  where  $P_\bullet$  is a principal  $G$ -bundle over  $\mathcal{G}_\bullet$ ,  $\theta$  is a connection on  $P_\bullet$  and  $\omega$  is a formal series of forms in  $\Omega_{dR}^*(\mathcal{G}_\bullet)$  such that

$$\Phi(\theta) = d\omega \bmod \mathcal{F}.$$

An isomorphism  $f : (P_\bullet, \theta, \omega) \rightarrow (P'_\bullet, \theta', \omega')$  between two multiplicative bundles over  $\mathcal{G}$  is an isomorphism  $f$  of the underlying bundles  $P_\bullet, P'_\bullet$  such that

$$\omega' - \omega = \Theta_1(\theta, f^* \theta') \bmod \tilde{\mathcal{F}}.$$

It follows as in [K2] that this defines an equivalence relation on multiplicative bundles, so we can make the following definition:

**Definition 3.4.** Let  $\mathcal{G}$  be a smooth étale groupoid. The set  $MK^\Phi(\mathcal{G}_\bullet; \mathcal{F})$  of isomorphism classes of  $(\mathcal{F}, \Phi)$ -multiplicative bundles is called the *multiplicative K-theory of  $\mathcal{G}$  with respect to  $(\mathcal{F}, \Phi)$* .

If there is no risk of ambiguity, we will omit  $\Phi$  and  $\mathcal{F}$  from the notation.

## 4 Characteristic classes for secondary theories associated to étale groupoids

Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Given a principal  $G$ -bundle with a connection  $\theta$  on the simplicial smooth manifold  $\mathcal{G}_\bullet$  associated to an étale groupoid  $\mathcal{G}$  and an invariant polynomial  $\Phi$  of homogeneous degree  $k$  we will associate characteristic classes with values in multiplicative cohomology groups

and in groups of differential characters of  $\mathcal{G}_\bullet$  associated to any filtration  $\mathcal{F}$  of the simplicial de Rham complex  $\Omega_{dR}^*(\mathcal{G}_\bullet)$ . This generalizes Karoubi's secondary characteristic classes as constructed in [K2] to smooth étale groupoids. An approach with a somehow similar flavour can also be found in [CM2].

Let  $\mathcal{F} = \{F^r \Omega_{dR}^*(\mathcal{G}_\bullet)\}$  be a filtration of the de Rham complex of  $\mathcal{G}$  and  $\Gamma = (P_\bullet, \theta, \eta)$  a  $(\mathcal{F}, \Phi)$ -multiplicative bundle over  $\mathcal{G}$ .

The connection  $\theta$  on the principal  $G$ -bundle  $\pi_\bullet$  is given as a 1-form

$$\theta \in \Omega_{dR}^1(P_\bullet) \otimes \mathfrak{g}.$$

The characteristic form of Theorem 1.7

$$\Phi(\theta) \in \Omega_{dR}^{2k}(\mathcal{G}_\bullet)$$

can also be seen as a family of forms

$$\Phi(\theta^{(n)}) \in \Omega_{dR}^{2k}(\Delta^n \times \mathcal{G}_n) \otimes \mathfrak{g}$$

satisfying the compatibility conditions.

Since  $\Gamma$  is a multiplicative bundle we have

$$\Phi(\theta) = d\eta + \omega$$

where the forms  $\eta$  and  $\omega$  are compatible sequences  $\eta = \{\eta^{(n)}\}$  and  $\omega = \{\omega^{(n)}\}$  of differential forms with  $\omega \in F^r \Omega_{dR}^{2k}(\mathcal{G}_\bullet)$  and  $\eta \in \Omega_{dR}^{2k-1}(\mathcal{G}_\bullet)$ .

The connection  $\theta$  is the pullback of a connection  $\theta_{U_\bullet\bullet}$  on  $U_\bullet\bullet$  by a map  $\Psi$  as in Theorem 1.8. Let  $\Lambda$  be a subring of the complex numbers  $\mathbb{C}$ , and assume that  $\Phi$  corresponds under the Chern-Weil map to a  $\Lambda$ -valued cohomology class.

For every  $n$  the inclusion  $\iota_n : B_{n\bullet} \rightarrow B_\bullet\bullet$  induces isomorphisms in cohomology since  $\|B_{n\bullet}\|$  is homotopy equivalent to the classifying space  $BG$  of  $G$ . We also have that  $\iota_n^* \theta_{U_\bullet\bullet} = \theta_{U_{n\bullet}}$  for every  $n$ . Since the form  $\Phi(\theta_{U_{n\bullet}})$  represents the class of  $\Phi$  by Theorem 1.7, and  $\iota_n^* \Phi(\theta_{U_\bullet\bullet}) = \Phi(\theta_{U_{n\bullet}})$ , we have that the form  $\Phi(\theta_{U_\bullet\bullet}) \in \Omega_{dR}^*(B_\bullet\bullet)$  represents the class of  $\Phi$ . Then it follows that there exist a compatible cocycle  $c \in C^{2k}(B_\bullet\bullet; \Lambda)$  and a compatible cochain  $v \in C^{2k-1}(B_\bullet\bullet; \mathbb{C})$  such that we have

$$\delta v = c - \Phi(\theta_{U_\bullet\bullet})$$

Since  $\Psi^*$  maps compatible cochains (in the bisimplicial sense) to compatible chains (in the simplicial sense), the triple  $\xi(\Gamma) = (\Psi^*(c), \omega, \Psi^*(v) + \eta)$  defines a cocycle in the cone complex

$$\text{cone}(C^*(\mathcal{G}_\bullet; \Lambda) \oplus F^r \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C}))$$

and since  $\omega$  is a form of degree  $2k$  also a cocycle in the cone complex

$$\text{cone}(C^*(\mathcal{G}_\bullet; \Lambda) \oplus \sigma_{\geq 2k} F^r \Omega_{dR}^*(\mathcal{G}_\bullet) \rightarrow C^*(\mathcal{G}_\bullet; \mathbb{C})).$$

The triple  $\xi(\Gamma)$  is a cocycle, because we have  $\delta \Psi^* c = \Psi^* \delta c = 0$ . Since  $c$  is a cocycle, we have that  $d\omega = d\Phi(\theta) + d^2\eta = 0$ , and also

$$\delta \Psi^* v = \Psi^* \delta v = \Psi^*(c - \Phi(\theta_{U_\bullet\bullet})) = \Psi^* c - \Phi(\theta) = \Psi^* c - (\omega + d\eta).$$

The class of  $\xi(\Gamma)$  is independent of the choices of  $c$  and  $v$ : If  $c'$  and  $v'$  are other choices, we must have  $c - c' = \delta u$  and  $\delta u = \delta(v - v')$ . Then, since  $H^{2k-1}(\|B_{\bullet\bullet}\|; \mathbb{C})$  is trivial, there exists a compatible cochain  $w$  such that  $\delta w = u + (v - v')$ . If  $\xi'(\Gamma)$  is the cocycle obtained from the different choice, then  $\xi(\Gamma) - \xi'(\Gamma) = (\Psi^* \delta u, 0, \Psi^*(v - v')) = d(\Psi^* u, 0, \Psi^* w)$ .

Hence for  $2r - m = 2k$  we can define the class of the multiplicative bundle  $(P_\bullet, \theta, \eta)$  in the multiplicative cohomology group  $MH_m^{2r}(\mathcal{G}_\bullet, \Lambda, \mathcal{F})$  to be the class of  $\xi(\Gamma)$ . Similarly the class of  $(P_\bullet, \theta, \eta)$  in  $\hat{H}_r^{2k-1}(\mathcal{G}_\bullet; \mathbb{C}/\Lambda; \mathcal{F})$  is the class of the triple  $\xi(\Gamma)$ .

**Proposition 4.1.** *The classes constructed above are characteristic classes of elements of  $MK^\Phi(\mathcal{G}_\bullet; \mathcal{F})$ .*

**Proof.** The naturality follows from the construction. We show that for two isomorphic multiplicative bundles  $\Gamma = (P_\bullet, \theta, \eta)$  and  $\Gamma' = (P'_\bullet, \theta', \eta')$  the cocycles  $\xi(\Gamma)$  and  $\xi(\Gamma')$  are cohomologous. We can assume  $P_\bullet = P'_\bullet$ , and write  $\Phi(\theta) = \omega + d\eta$  and  $\Phi(\theta') = \omega' + d\eta'$  with  $\omega, \omega' \in F^r \Omega_{dR}^{2k}(\mathcal{G}_\bullet)$ . Since the two multiplicative bundles are isomorphic we have

$$\eta' - \eta = \Theta_1(\Phi; \theta, \theta') + \sigma + d\rho$$

with  $\sigma \in F^r \Omega_{dR}^{2k-1}(\mathcal{G}_\bullet)$ . It follows that

$$\omega' - \omega = d(\Theta_1(\Phi; \theta, \theta') - (\eta' - \eta)) = -d(\sigma + d\rho).$$

Let  $\Psi'$  be the map pulling back  $\Gamma'$  given by Theorem 1.8 and let  $c', v'$  be the cochains used in the construction for the characteristic cycle  $\xi(\Gamma')$ . Then

$$\xi(\Gamma') - \xi(\Gamma) = (\Psi'^* c' - \Psi^* c, \omega' - \omega, \Psi'^* v' - \Psi^* v + \Theta_1(\Phi; \theta, \theta') + \sigma + d\rho)$$

is cohomologous to the triple  $\zeta = (\Psi'^* c' - \Psi^* c, 0, \Psi'^* v' - \Psi^* v + \Theta_1(\Phi; \theta, \theta'))$  since the two differ only by the coboundary of  $(0, -\sigma, \rho)$ . We can choose  $c' = c$  and  $v' = v + \Theta_1(\Phi; \theta_{U\bullet\bullet}, \theta'_{U\bullet\bullet})$ , where  $\theta'_{U\bullet\bullet}$  is the connection pulling back to  $\theta'$  under  $\Psi'^*$  given by Theorem 1.8. Hence we have, using the naturality of the first transgression form,

$$\zeta = (\Psi'^* c - \Psi^* c, 0, \Psi'^* v - \Psi^* v + \Theta_1(\Phi; \Psi'^* \theta_{U\bullet\bullet}, \theta') + \Theta_1(\Phi; \theta, \theta')).$$

Proposition 3.2 now implies that

$$\begin{aligned} \Theta_1(\Phi; \Psi^* \theta_{U\bullet\bullet}, \Psi'^* \theta_{U\bullet\bullet}) + d\Theta_2(\Phi; \Psi^* \theta_{U\bullet\bullet}, \Psi'^* \theta_{U\bullet\bullet}, \theta') = \\ \Theta_1(\Phi; \Psi'^* \theta_{U\bullet\bullet}, \theta') + \Theta_1(\Phi; \Psi^* \theta_{U\bullet\bullet}, \theta'). \end{aligned}$$

But  $\Psi'$  and  $\Psi$  are homotopic, so there is a chain homotopy  $H$  between the induced cochain maps and using  $H$  we can write  $\zeta$  as

$$(\delta Hc, 0, \delta Hv + H\delta v + \Theta_1(\Phi; \Psi^* \theta_{U\bullet\bullet}, \Psi'^* \theta_{U\bullet\bullet}) + d\Theta_2(\Phi; \Psi^* \theta_{U\bullet\bullet}, \Psi'^* \theta_{U\bullet\bullet}, \theta')).$$

Therefore  $\zeta$  is cohomologous to  $(\delta Hc, 0, H\delta v + \Theta_1(\Phi; \Psi^* \theta_{U\bullet\bullet}, \Psi'^* \theta_{U\bullet\bullet}))$ , and since the transgression forms  $\Theta_1(\cdot; \cdot, \cdot)$  are compatible with chain homotopies (see [DHZ], appendix A), the former cocycle is cohomologous to

$$(\delta Hc, 0, H\delta v + H\Phi(\theta_{U\bullet\bullet})) = d(Hc, 0, 0)$$

because  $Hc = H(\delta v + \Phi(\theta_{U\bullet\bullet}))$ .  $\square$



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